A FURTHER VARIATION OF THE BANACH-MAZUR GAME AND FORCING AXIOMS (Iterated Forcing Theory and Cardinal Invariants)

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ABSTRACT. In this short note we quickly introduce a class of posets defined in terms of a variation of the generalized Banach-Mazur game, and state a theorem about the extent of preservation of forcing axioms under forcing over posets in that class. Full proofs and other results in this subject will be contained in our paper in preparation. This work is based on an early joint work with Bernhard König.

In [1], König and the author proved the following theorem.

Theorem 1 (König–Y. [1]). PFA is preserved under any ω_2-closed forcing.

One way to generalize this theorem is to find a property of posets weaker than the ω_2-closedness, such that forcing over any poset with the property still preserves PFA.

One well-known weakening of the closedness properties of posets is the *strategic closedness* properties, defined in terms of the generalized Banach-Mazur game.

Definition 2. For a poset $\mathbb{P}$ and an ordinal α, the generalized Banach-Mazur game $G_\alpha(\mathbb{P})$ is a two-player game played as follows: A play of this game are developed in at most α innings. At the γ-th inning ($\gamma < \alpha$), if γ is nonlimit (successor or zero) Player I makes a move first and then Player II makes a move, and if γ is limit only Player II makes a move. Each move of both players must be a $\mathbb{P}$-condition stronger than all preceding moves. Player II wins if she completed α innings without getting unable to make a legal move on the way.

$$\begin{array}{lcccccccccc} \mathrm{I}: & a_0 & & a_1 & & \cdots & & & a_{\omega+1} & & \cdots \\ \mathrm{II}: & & b_0 & & b_1 & \cdots & b_\omega & & & b_{\omega+1} & \cdots \end{array}$$

$\mathbb{P}$ is said to be *α-strategically closed* if Player II has a winning strategy (in the obvious sense) for $G_\alpha(\mathbb{P})$.

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Note that it is clear that every ω_2-closed poset is (ω_1+1)-strategically closed.

Unfortunately, it is known that the (ω_1+1)-strategic closedness is not enough to preserve PFA: In fact, the natural poset to force $\square_{\omega_1}$ is (ω_1+1)-strategically closed, whereas $\square_{\omega_1}$ fails under PFA.

In [2], the author introduced a new property of posets, whose strength lies between those of the ω_2-closedness and the (ω_1+1)-strategic closedness, and proved that PFA is preserved under any forcing over posets with this property. This property is defined in terms of the following variation of the generalized Banach-Mazur game.

Definition 3. For a separative poset $\mathbb{P}$, $G^*(\mathbb{P})$ denotes the following two-player game: Innings of a play of the game are indexed by countable ordinals. At the α-th inning for each $\alpha<\omega_1$, Player I chooses a countable compatible subset A_α of $\mathbb{P}$ and then Player II chooses $b_\alpha\in\mathbb{P}$.

$$\begin{array}{lcccccccccc} \mathrm{I}: & A_0 & & A_1 & & \cdots & A_\omega & & A_{\omega+1} & & \cdots \\ \mathrm{II}: & & b_0 & & b_1 & \cdots & & b_\omega & & b_{\omega+1} & \cdots \end{array}$$

Players must obey the following requirements: For each $\alpha<\omega_1$,

(a) b_α extends all $\mathbb{P}$-conditions in A_α,
(b) $A_{\alpha+1}\supseteq A_\alpha$,
(c) $\inf A_{\alpha+1}\leq_{\mathcal{B}(\mathbb{P})} b_\alpha$ (where $\mathcal{B}(\mathbb{P})$ denotes the Boolean completion of $\mathbb{P}$, and the infimum in the left-hand-side is computed in $\mathcal{B}(\mathbb{P})$) and
(d) $A_\alpha=\bigcup_{\gamma<\alpha}A_\gamma$ if α is a limit ordinal, .

Player II wins if she was able to make all ω_1 moves without making Player I unable to make a legal move on the way, AND $\{b_\alpha\mid\alpha<\omega_1\}$ has a common extension.

Note that by replacing each move of Player I by its Boolean infimum, a play of $G^*(\mathbb{P})$ can be seen as a play of $G_{\omega+1}(\mathcal{B}(\mathbb{P}))$ (note also that each move of Player I at limit innings in $G^*(\mathbb{P})$ is automatically determined from preceding moves and thus is ignorable). In fact, the existence of a winning strategy of Player II for $G^*(\mathbb{P})$ and that for $G_{\omega+1}(\mathbb{P})$ are equivalent. The introduction of G^*-games makes sense when we consider a strong form of winning strategies.

Definition 4. For a separative poset $\mathbb{P}$, a $*$-tactic for $\mathbb{P}$ is a function $\tau:[\mathbb{P}]^{\leq\omega}\to\mathbb{P}$. In a play of $G^*(\mathbb{P})$, Player II is said to play by a $*$-tactic τ if she choose $\tau(A_\alpha)$ at the α-th inning for each $\alpha<\omega_1$, responding the opponent's α-th move A_α. A $*$-tactic τ is said to be a winning one if Player II wins $G^*(\mathbb{P})$ whenever she plays by τ. $\mathbb{P}$ is said to be *$*$-tactically closed* if $\mathbb{P}$ has a winning $*$-tactic.

Theorem 5 (Y. [2]). PFA is preserved under any $*$-tactically closed forcing.

Now we introduce a further variation of the Banach-Mazur game.

Definition 6. For a separative poset $\mathbb{P}$, $G^{**}(\mathbb{P})$ denotes the game similar to $G^{*}(\mathbb{P})$, with the only difference that, for each $\alpha < \omega_1$, Player I must obey the following additional rule, besides of (a)–(d) in Definition 3:

(e) $p \leq_{\mathbb{P}} b_\alpha$ for each $p \in A_{\alpha+1} \setminus A_\alpha$.

Note that, as far as players play with perfect information, $G^{*}(\mathbb{P})$ and $G^{**}(\mathbb{P})$ are again essentially the same game, because at each turn in a play of $G^{*}(\mathbb{P})$, Player I can rearrange his move to a stronger one satisfying (e). This seemingly small change, however, makes a remarkable difference when considering $*$-tactics.

Definition 7. $\mathbb{P}$ is said to be *$**$-tactically closed* if there exists a $*$-tactic τ for $\mathbb{P}$ such that Player II wins $G^{**}(\mathbb{P})$ whenever she plays by τ.

Definition 8. $\mathrm{SCP}^-_{\mathrm{e}}$ denotes the following statement:

There exists a sequence $\langle C_\alpha \mid \alpha \in S^2_0 \rangle$ such that

(1) For every $\alpha \in S^2_0$, C_α is a countable unbounded subset of α.
(2) For every $\beta \in S^2_1$, there exists a closed unbounded subset C of $\beta \cap S^2_0$ with $\mathrm{o.t.}(C) = \omega_1$, such that $C_{\alpha'} \cap \alpha = C_\alpha$ holds for every $\alpha, \alpha' \in C$ with $\alpha < \alpha'$.

Theorem 9. (1) $\mathrm{SCP}^-_{\mathrm{e}}$ fails under PFA.
(2) There exists a $**$-tactically closed forcing which forces $\mathrm{SCP}^-_{\mathrm{e}}$.

Note that Theorem 9 tells that PFA is not necessarily preserved under forcing over $**$-tactically closed posets, unlike $*$-tactically closed ones.

Then how badly may $**$-tactically closed forcing destroy PFA? As an earlier result on this line, König and the author observed the following.

Theorem 10 (König–Y.(2013)). Assume MM. Then after forcing over the natural $**$-tactically closed poset forcing $\mathrm{SCP}^-_{\mathrm{e}}$, ω_2 remains to have the tree property (therefore $\square_{\omega_1}$ remains to fail, for example).

Extending Theorem 10 now we have the following.

Theorem 11. Assume PFA. Then for any $**$-tactically closed poset $\mathbb{P}$, it holds that

$$\Vdash_{\mathbb{P}} \mathrm{MA}_{\omega_1}(\sigma\text{-closed} * \mathrm{ccc}).$$

References

[1] Bernhard König and Yasuo Yoshinobu. Fragments of Martin's Maximum in generic extensions. *Mathematical Logic Quarterly*, 50:297–302, 2004.

[2] Yasuo Yoshinobu. The *-variation of banach-mazur game and forcing axioms. *Annals of Pure and Applied logic*, 168(6):1335–1359, 2017.

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